

CONSTRAINING MAPPING CLASS GROUP HOMOMORPHISMS USING FINITE SUBGROUPS

LEI CHEN AND JUSTIN LANIER

ABSTRACT. We give a new proof of a result of Aramayona–Souto that constrains homomorphisms between mapping class groups of closed surfaces. The proof involves an analysis of finite subgroups. Using the same approach, we also prove that for $g \geq 3$ every homomorphism from $\text{Mod}(S_g)$ to $\text{Homeo}^+(\mathbb{S}^2)$ is trivial. This extends a result of Franks–Handel, who showed the same for homomorphisms to $\text{Diff}(\mathbb{S}^2)$ with $g \geq 7$. We also show that for $g \geq 3$ every homomorphism from $\text{Mod}(S_g)$ to $\text{Homeo}^+(\mathbb{R}^3)$ is trivial, and that the same holds for homomorphisms to $\text{Homeo}^+(\mathbb{S}^3)$ and to $\text{Diff}^+(\mathbb{S}^4)$ when $g \geq 42$.

0. INTRODUCTION

Let S_g be the connected, closed, orientable surface of genus g and let \mathbb{S}^n be the n -sphere. The groups $\text{Homeo}^+(X)$ and $\text{Diff}^+(X)$ are, respectively, the groups of orientation-preserving homeomorphisms and diffeomorphisms of an orientable manifold X . The mapping class group $\text{Mod}(X)$ is $\pi_0(\text{Homeo}^+(X))$, the group of isotopy classes of orientation-preserving homeomorphisms of X .

In this paper we analyze torsion to prove theorems about homomorphisms from $\text{Mod}(S_g)$ to mapping class groups and to homeomorphism and diffeomorphism groups of spheres. Much has been learned about homomorphisms of mapping class groups by analyzing torsion. Several examples appear later in the introduction, and other examples include work by Markovic, Mann–Wolff, and the first author [20, 19, 7]. Our approach here is similar in spirit, but for the problems we consider it is not sufficient to analyze individual periodic elements in isolation. We explain this obstacle below. As a consequence, we proceed instead by analyzing non-cyclic finite subgroups of $\text{Mod}(S_g)$. Our theorems are as follows.

Theorem 1. *For $g \geq 3$ and $0 \leq h < 2g - 1$ with $h \neq g$, every homomorphism $\phi : \text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$ is trivial. When $g \geq 3$ is odd, the same conclusion holds for the bounds $0 \leq h < 2g + 1$ with $h \neq g$.*

Theorem 2. *For $g \geq 3$, every homomorphism from $\text{Mod}(S_g)$ to either $\text{Homeo}^+(\mathbb{S}^2)$ or $\text{Homeo}^+(\mathbb{R}^3)$ is trivial. For $g \geq 42$, every homomorphism from $\text{Mod}(S_g)$ to either $\text{Homeo}^+(\mathbb{S}^3)$ or $\text{Diff}^+(\mathbb{S}^4)$ is trivial.*

Aramayona–Souto proved a rigidity theorem about homomorphisms between mapping class groups [2]. Theorem 1 recovers their theorem in the case of closed surfaces of different genus. Our proof also yields a small improvement in their genus bounds. Theorem 2 is a strengthening and extension of a theorem of Franks–Handel, who consider maps from $\text{Mod}(S_g)$ to $\text{Diff}(\mathbb{S}^2)$ [12]. We give a more detailed comparison of our results and approaches later in the introduction.

Our proofs of Theorems 1 and 2 use the following result of the second author and Margalit. A group element is said to normally generate if its normal closure equals the group.

Theorem 3 (Theorem 1.1, [17]). *For $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates $\text{Mod}(S_g)$.*

It follows immediately from this theorem that any homomorphism from $\text{Mod}(S_g)$ that has a nontrivial nonhyperelliptic periodic element in its kernel is trivial. The strong constraints on homomorphisms provided by Theorem 3 are not, however, immediately applicable to the settings of Theorems 1 and 2. Under the hypotheses of each of these theorems, there are cases where, a priori, there could be a homomorphism to the target group where no nontrivial periodic mapping class of $\text{Mod}(S_g)$ would lie in the kernel. For instance, for every periodic element in $\text{Mod}(S_7)$, there is an element of $\text{Mod}(S_9)$ of the same order. For Theorem 2, the situation is more extreme, as $\text{Homeo}^+(\mathbb{S}^2)$ contains elements of every finite order. Despite these obstacles, we are still able to apply Theorem 3 to these situations by analyzing non-cyclic finite subgroups.

Homomorphisms between mapping class groups.

Aramayona–Souto proved a theorem constraining homomorphisms between the mapping class groups of surfaces $S_{g,n,b}$, which are surfaces of genus g with n punctures and b boundary components [2]. The mapping class groups they consider are pure, in that they require mapping classes to fix punctures and boundary components pointwise.

Theorem 4 (Aramayona–Souto, Theorem 1.1, [2]). *Let $S = S_{g,n,b}$ and $S' = S'_{g',n',b'}$, such that $g \geq 6$ and $g' \leq 2g - 1$. If $g' = 2g - 1$, suppose also that S' is not closed. Then every nontrivial homomorphism $\phi : \text{Mod}(S) \rightarrow \text{Mod}(S')$ is induced by an embedding $S \rightarrow S'$.*

Aramayona–Souto also prove that the conclusion of this theorem holds when $g = g' \in \{4, 5\}$. They explain the necessity of their upper bound of $2g - 1$ by observing that there is a “double embedding” homomorphism $\text{Mod}(S_{g,0,1}) \rightarrow \text{Mod}(S_{2g,0,0})$.

When restricted to the case of closed surfaces, Theorem 4 says that for $g \geq 6$ and $h < 2g - 1$, every homomorphism $\phi : \text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$ is trivial, except for the possibility of an isomorphism in the case $g = h$. Some cases of this result for closed surfaces were previously known. The range $h < g$ was previously handled by Harvey–Korkmaz [13], while the case $h = g$ was treated by Ivanov–McCarthy under the further hypothesis that the homomorphism is injective [14].

Theorem 1 gives a new proof of Theorem 4 when restricted to the case where S and S' are closed surfaces and $g \neq g'$. Further, it is an easy consequence of the proof of Theorem 1 that we may replace S_h with a surface of genus h and arbitrarily many punctures and boundary components, since $\text{Mod}(S_{h,n,b})$ has no additional finite subgroups compared to $\text{Mod}(S_h)$. For this replacement, note that it is not necessary for us to restrict to $\text{Mod}(S_{h,n,b})$ being pure. Our Theorem 1 also covers the additional small values of g of 3, 4, and 5, confirming an expectation that Aramayona–Souto state in their paper. We also extend their upper bounds for g' slightly when g is odd. This is possible because, for closed surfaces, their upper bound of $2g - 1$ does not have the same natural justification that appears for surfaces with boundary.

It is important to note that, even for closed surfaces, some upper bound is necessary on h to ensure that the homomorphism is trivial. First, since $\text{Mod}(S_g)$ is residually finite, it has a rich supply of finite quotients. As every finite group is a subgroup

for some $\text{Mod}(S_h)$, we obtain for all $g > 0$ non-trivial homomorphisms $\text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$ that factor through finite quotients. An even more striking reason why some upper bound on h is necessary is a result of Aramayona–Leininger–Souto: that for all $g \geq 2$, there exists a nontrivial connected cover of S_g such that $\text{Mod}(S_g)$ injects into the mapping class group of the cover [1].

This raises the following natural question:

Question 5. *For each $g \geq 3$, what is the smallest $h > g$ such that there exists a nontrivial homomorphism $\phi : \text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$?*

Homomorphisms to homeomorphism groups of spheres.

Franks–Handel prove a number of theorems showing that homomorphisms from mapping class groups to homeomorphism and diffeomorphism groups are trivial [12]. Their main result is that homomorphisms $\text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C})$ are trivial for S a finite-type surface whenever $g \geq 3$ and $n < 2g$. They also apply the theorem of Aramayona–Souto to show that, with the same genus bounds and the added requirement that $g' > 1$, every homomorphism $\text{Mod}(S_g) \rightarrow \text{Homeo}(S_{g'})$ is trivial. As these bounds exclude $g' = 0, 1$, they go on to show that for $g \geq 3$, all homomorphisms from $\text{Mod}(S_g)$ to each of $\text{Homeo}(\mathbb{S}^1)$ and $\text{Homeo}(S_1)$ are trivial, and they also show the following theorem.

Theorem 6 (Theorem 1.4, [12]). *For $g \geq 7$, every homomorphism $\phi : \text{Mod}^\pm(S_g) \rightarrow \text{Diff}(\mathbb{S}^2)$ is trivial.*

Our Theorem 2 includes an extension of this theorem of Franks–Handel to several small genus cases. Note that the lower bound of $g \geq 3$ in the statement of Theorem 2 is necessary, since in the cases $g = 1, 2$ there are nontrivial homomorphisms that factor through the abelianization of $\text{Mod}(S_g)$.

In Theorem 2, one of our target groups is a diffeomorphism group rather than a homeomorphism group. Unlike in the case of $\text{Homeo}^+(\mathbb{S}^2)$, for higher-dimensional spheres there exist actions by finite groups that have wildly embedded fixed point sets, and such actions cannot be smooth; see the survey article by Zimmermann for a discussion [31]. Recent work of Pardon shows that there do not exist any isomorphism types of finite subgroups of $\text{Homeo}^+(\mathbb{S}^3)$ that do not also occur in $\text{Diff}^+(\mathbb{S}^3)$ [26]. On the other hand, the finite subgroups of $\text{Homeo}^+(\mathbb{S}^4)$ are not yet classified. Under the further hypothesis of smoothness, however, there are results classifying finite group actions on \mathbb{S}^4 ; we use these in our proof of Theorem 2.

We close with two questions. Zimmermann observes in the same survey article that just by considering faithful, real, linear representations of finite groups, it is clear that every finite group appears as a subgroup of $\text{Diff}^+(\mathbb{S}^d)$ for some d . Therefore arguments using finite subgroups are likely not sufficient to constrain homomorphisms to these groups.

Question 7. *For $g \geq 3$ and $d \geq 5$, when are there nontrivial homomorphisms $\phi : \text{Mod}(S_g) \rightarrow \text{Diff}^+(\mathbb{S}^d)$?*

On the other hand, $\text{Mod}(S_g)$ does act nontrivially—and, in fact, faithfully—by homeomorphisms on $\mathcal{PMF}(S_g) \cong \mathbb{S}^{6g-7}$. By a suspension construction, $\text{Mod}(S_g)$ acts faithfully on each sphere \mathbb{S}^d , $d \geq 6g - 7$. We therefore ask the following:

Question 8. *For $g \geq 3$ and $3 \leq d < 6g - 7$, are there any nontrivial homomorphisms $\phi : \text{Mod}(S_g) \rightarrow \text{Homeo}^+(\mathbb{S}^d)$?*

Outline. After proving some preliminary lemmas, we prove Theorem 1 in Section 1. We then prove Theorem 2 in Section 2.

Acknowledgments. We thank Dan Margalit for a number of helpful conversations and for comments on a draft of this article. We thank Samuel Taylor for suggesting that we consider maps to homeomorphism groups of higher-dimensional spheres. JL is supported by NSF Grant DGE - 1650044.

1. HOMOMORPHISMS BETWEEN MAPPING CLASS GROUPS

The strategy for proving Theorem 1 is straightforward. We first show that some periodic element is in the kernel of the given homomorphism $\text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$. We then show that this implies that the homomorphism is trivial. We lay the groundwork for these two steps in two lemmas, Lemmas 9 and 10. We then prove Theorem 1.

We first show that a nontrivial periodic element must lie in the kernel of the homomorphism, for the simple reason that there exists a finite subgroup of $\text{Mod}(S_g)$ that does not exist in any $\text{Mod}(S_h)$ for h in the specified range. The finite subgroups that we use lie in two infinite families, one for when g is even, the other for when g is odd. These families of subgroups were studied by May–Zimmerman; we will draw out the salient features of their work and will repeat some of their arguments for the sake of clarity, but we refer the reader to their papers for full details.

For $n \geq 2$, let DC_n be the dicyclic group of order $4n$, given by the presentation

$$\langle x, y \mid x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1} \rangle.$$

May–Zimmerman showed that when n is even, DC_n has *strong symmetric genus* n ; that is, the first genus g for which DC_n appears as a subgroup of $\text{Mod}(S_g)$ is when $g = n$ [21, Theorem 1]. Similarly, they show that when n is odd, $C_4 \times D_n$ has strong symmetric genus n , where C_n is the cyclic group of order n and D_n is the dihedral group of order $2n$ [22, Theorem 3]. Let G stand for any one of these finite groups. In proving their results, May–Zimmerman first show that G does in fact appear as a subgroup of the specified mapping class group $\text{Mod}(S_g)$. To guarantee that this is the first appearance, they then give lower bounds on h for any other $\text{Mod}(S_h)$ containing G as a subgroup, showing that $h > g$. In the proof of Lemma 9, we follow their method and improve their lower bounds on h .

Lemma 9. *When $g \geq 2$ is even, DC_g appears as a subgroup of $\text{Mod}(S_g)$ and does not appear in any other $\text{Mod}(S_h)$ with $h < 2g - 1$. When $g \geq 3$ is odd, $C_4 \times D_g$ appears as a subgroup of $\text{Mod}(S_g)$ and does not appear in any other $\text{Mod}(S_h)$ with $h < 2g + 1$.*

Before proving Lemma 9, we require some preliminaries; see the article of Broughton as a reference [4]. Recall that for any faithful orientation-preserving action of a finite group G on a hyperbolic surface S_g by isometries, we have that

$$(1) \quad A = 2g_0 - 2 + \sum_{i=1}^r \left(1 - \frac{1}{\lambda_i}\right)$$

where the normalized area A is the hyperbolic area of the quotient orbifold scaled by $\frac{1}{2\pi}$, the g_0 is the genus of the quotient orbifold, the λ_i are the orders of the cone points, and the r is the number of cone points. The data of this action is called its

signature and is often encoded as $(g_0; \lambda_1, \dots, \lambda_r)$. The normalized area, group order, and genus g of the original surfaces are related by the Riemann–Hurwitz equation

$$|G| \cdot A = 2g - 2.$$

Finally, for a signature of $(g_0; \lambda_1, \dots, \lambda_r)$ to arise from an action of G , these values must satisfy the Riemann–Hurwitz equation and there must be elements $a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_r \in G$ that together generate G , where $|c_i| = \lambda_i$, and that satisfy

$$(2) \quad \prod_{i=1}^g [a_i, b_i] \cdot \prod_{j=1}^r c_j = 1.$$

This equation follows from the fact that a finite group acting on a hyperbolic surface must arise as a quotient of a Fuchsian group.

This main ideas of constraining finite group actions on surfaces are similar to the proofs of the classical $84(g-1)$ and $4g+2$ theorems; see, for instance, [10, Theorems 7.4, 7.5].

Proof of Lemma 9. May–Zimmerman argue that when $g \geq 2$ is even and DC_g is a subgroup of $\text{Mod}(S_h)$, either $h = g$ or $h > g$. We will sharpen their second bound to show that in fact either $h = g$ or $h \geq 2g - 1$. Their arguments run by showing a dichotomy that either the normalized area A of a fundamental domain of the action of DC_g on S_h satisfies either $A = \frac{1}{2} - \frac{1}{2g}$ or else $A \geq \frac{1}{2}$. The value of h can then be computed from the Riemann–Hurwitz equation: $h = 1 + 2gA$. We will improve the latter bound to $A \geq \frac{g-1}{g}$, which implies that $h \geq 2g - 1$.

We consider cases based on the values of g_0 and r . By Equation (1), if $g_0 \geq 2$, then $A \geq 2$. Next, if $g_0 = 1$, then $r \geq 1$ since g is assumed to be at least 2. If $g_0 = 1$ and $r \geq 2$, then $A \geq 1$. If $g_0 = 1$ and $r = 1$, then DC_g has a generating set $\langle a, b \rangle$ where $[a, b] = c^{-1}$ and $|c| = \lambda_1$. It is straightforward to check that $|[a, b]| = g$ for any generating pair for DC_g . This yields $A \geq \frac{g-1}{g}$.

Finally, consider the case $g_0 = 0$. Since $A > 0$, we know that $r \geq 3$. If $r \geq 5$, then we have $A \geq \frac{3}{4} \cdot 2 + \frac{1}{2} \cdot 3 - 2 = 1$. Now assume that $r \leq 4$. There is a generating set of DC_g satisfying $\prod_{j=1}^r c_j = 1$. Any generating set for DC_g contains at least one generator outside of $\langle x \rangle$, and by the product restriction there are an even number of these in one of our generating sets. Further, each element outside of $\langle x \rangle$ has order 4. We now treat the cases $r = 3$ and $r = 4$ in turn.

Let $r = 3$. Then exactly two c_i lie outside of $\langle x \rangle$ and these have order 4. These must generate DC_g , and a short computation shows that their product has order $2g$. Therefore if $r = 3$, the signature is $(0; 4, 4, 2g)$, $A = \frac{1}{2} - \frac{1}{2g}$, and $h = g$, as shown by May–Zimmerman.

Finally, let $r = 4$. There are either two or four λ_i equal to 4 that correspond to elements outside of $\langle x \rangle$. If there are four, then $A = 1$. If there are two and the corresponding c_i together form a generating pair for DC_g , then the order of their product is relatively prime to $2g$. Then the product of the remaining c_i must equal the inverse of this product and so has the same order. If one of the remaining λ_i is either 2 or 3, then the last λ_i must be $2g$ or $2g/3$, respectively. (The latter case is only possible when g is a multiple of 3.) These yield $A = \frac{2g-1}{2g} > \frac{g-1}{g}$ and $A = \frac{7}{6} - \frac{3}{2g}$. The latter is at least 1 when $g \geq 12$ and also satisfies the necessary bound when $g = 6$.

The final possibility is that there are exactly two λ_i equal to 4 that correspond to elements outside of $\langle x \rangle$ and so that the corresponding c_i together do not form a generating pair for DC_g . Again, the product of the remaining c_i must equal the inverse of this product and so have the same order. Since a product in these remaining two c_i must generate $\langle x \rangle$, if one of the remaining λ_i is either 2 or 3, then the last λ_i must be again be $2g$ or $2g/3$, respectively. The result follows.

Similarly, May–Zimmerman argue that when $g \geq 3$ is odd and $C_4 \times D_g$ is a subgroup of $\text{Mod}(S_h)$, either $h = g$ or $h > g$. We will show how their arguments imply that $h = g$ or $h \geq 2g + 1$. Their proof proceeds by showing that the normalized area A of a fundamental domain of the action on S_h by $C_4 \times D_g$ is either exactly $\frac{1}{4} - \frac{1}{4g}$ or else $A \geq \frac{1}{2}$, leaving a few of the final cases as an exercise. These bounds imply that either $h = 1 + 4g(\frac{1}{4} - \frac{1}{4g}) = g$ or $h \geq 1 + 4g \cdot \frac{1}{2} = 2g + 1$ whenever $C_4 \times D_g$ is a subgroup of $\text{Mod}(S_h)$, as desired.

In what follows, we recap the arguments of May–Zimmerman and fill in the cases they leave as an exercise. If $g_0 \geq 1$, then $A \geq \frac{1}{2}$. Further, if $g_0 = 0$, then $r \geq 3$, and also the λ_i are all even. If $r \geq 4$, then $A \geq \frac{1}{2}$. If $r = 3$, they argue that if $\lambda_1 = 2$ then $h = g$, and if $\lambda_1 \geq 6$ then $A \geq \frac{1}{2}$. It remains to treat $\lambda_1 = 4$. For $\lambda_2 \geq 8$, they show that $A \geq \frac{1}{2}$; they leave $\lambda_2 = 4, 6$ as an exercise.

If $\lambda_1 = 4$ and $\lambda_2 = 4$, then each generator c_1, c_2 must be a generator of C_4 times a reflection in D_n , and necessarily the second factor of $c_1 \cdot c_2$ must be a rotation of order n . But a pair of such elements does not generate $C_4 \times D_g$; instead they generate an index 2 normal subgroup corresponding to a quotient to C_2 given by the parity of word length.

Similarly, if $\lambda_1 = 4$ and $\lambda_2 = 6$, then c_1 must be a generator of C_4 times a reflection in D_n , and c_2 must be either the identity or the square of a generator in the first factor and a rotation of order 3 in the second factor. This last condition is impossible unless n is a multiple of 3. But a rotation that does not generate the rotation subgroup in D_n cannot be a member of a generating pair for D_n . So this case is impossible as well. \square

We now prepare the second step, showing that a periodic element in the kernel of $\text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$ implies that the homomorphism is trivial whenever h is in the specified range, even in the case when it contains a hyperelliptic involution. The following lemma is similar to a result proved and applied by Harvey–Korkmaz to show their result on homomorphisms $\text{Mod}(S_g) \rightarrow \text{Mod}(S_h)$ where $g > h$ [13, Theorem 7]. Lemma 10 has the advantage of giving a uniform treatment for all $g \geq 3$.

Lemma 10. *Let $g \geq 3$ and let $\psi : \text{Sp}(S_g) \rightarrow G$ be a homomorphism with a nontrivial periodic element in its kernel. If G does not contain $(\mathbb{Z}/3)^g$ as a subgroup, then ψ is trivial.*

Proof. We proceed by constructing normal generators of $\text{Sp}(2g, \mathbb{Z})$ and showing that at least one must be in the kernel of ψ .

The group $\text{Sp}(2g, \mathbb{Z})$ contains $M = (C_3)^g$ as a subgroup, with generators m_1 to m_g of the form

$$m_i = \begin{bmatrix} I_{2i-2} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{2g-2i-4} \end{bmatrix}$$

where A is the 2×2 matrix

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

and I_n denotes the $n \times n$ identity matrix. Every nontrivial element in M is a normal generator of $\mathrm{Sp}(2g, \mathbb{Z})$, since it is the image of a normal generator of $\mathrm{Mod}(S_g)$, as we now show. The desired elements of $\mathrm{Mod}(S_g)$ are products of roots of Dehn twists about separating curves illustrated in Figure 1. For any nontrivial element $m \in M$, there is a corresponding mapping class \tilde{m} and a nonseparating curve c so that c and $\tilde{m}(c)$ intersect exactly once. By the well-suited curve criterion, it follows that the mapping class $\tilde{m}(c)$ is a normal generator of $\mathrm{Mod}(S_g)$ [16, Lemma 2.2]; so too is its image in $\mathrm{Sp}(2g, \mathbb{Z})$, since normal generators descend to quotients.

As G contains no subgroup isomorphic to M , some normal generator of $\mathrm{Sp}(2g, \mathbb{Z})$ lies in the kernel of ψ ; therefore ψ is trivial. \square

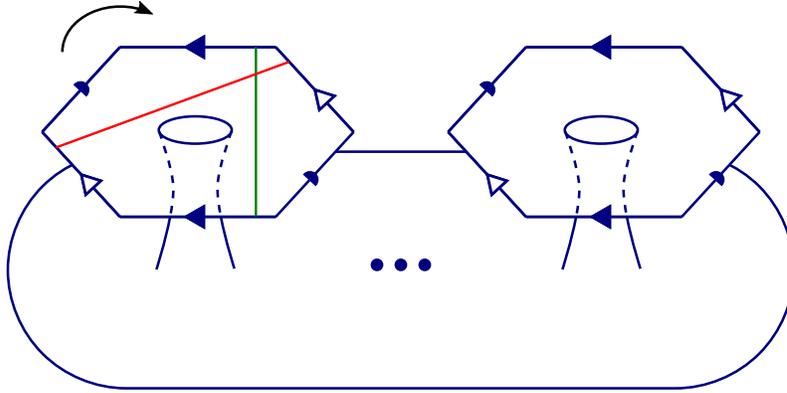


FIGURE 1. The surface S_g , where each of g hexagons with sides identified as indicated yields a handle. Each nontrivial element $\tilde{m} \in \tilde{M}$ consists of $1/3$ or $2/3$ rotations about some of the hexagons; we have $i(c, \tilde{m}(c)) = 1$ for some nonseparating simple closed curve c .

The following corollary can be thought of as an amplification of a theorem of Farb–Masur in the case of $\mathrm{Sp}(2g, \mathbb{Z})$; they show that for any irreducible lattice Γ in a semisimple lie group G of \mathbb{R} -rank at least two, the image of any homomorphism $\phi : \Gamma \rightarrow \mathrm{Mod}(S)$ is finite [11, Theorem 1.1].

Corollary 11. *Let $g \geq 3$ and let $h \leq 3^{g-1}$. Then every homomorphism $\psi : \mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{Mod}(S_h)$ is trivial.*

Proof. This follows immediately from Lemma 10 by a result of Müller–Sarkar, who showed that the strong symmetric genus of $(C_3)^g$ is $1 + 3^{g-1} \cdot \mu_0(g)$, where $\mu_0(g) \geq 1$ when $g \geq 3$ [25, Section 9.1]. \square

With these preliminaries established, the proof of Theorem 1 is straightforward.

Proof of Theorem 1. Let $g \geq 3$ be even and $g < h < 2g - 1$. By Lemma 9, $\mathrm{Mod}(S_g)$ contains DC_g as a subgroup and $\mathrm{Mod}(S_h)$ does not. Then for any homomorphism $\phi : \mathrm{Mod}(S_g) \rightarrow \mathrm{Mod}(S_h)$, a nontrivial periodic element f lies in the kernel. If f is not a hyperelliptic involution, we conclude that ϕ is trivial by Theorem 3. If f is a

hyperelliptic involution, then ϕ factors through the symplectic representation. Since $h < 2g - 1 \leq 3^{g-1}$, we conclude that ϕ is trivial by Corollary 11.

The proof for odd $g \geq 3$ proceeds in the same way using the finite group $C_4 \times D_g$. \square

2. HOMOMORPHISMS TO HOMEOMORPHISM GROUPS OF SPHERES

In this section we prove Theorem 2. We consider each of the four target groups in turn: $\text{Homeo}^+(\mathbb{S}^2)$, $\text{Homeo}^+(\mathbb{R}^3)$, $\text{Homeo}^+(\mathbb{S}^3)$, and then $\text{Diff}^+(\mathbb{S}^4)$. The constraining finite subgroups are the same for the first two target groups; when the target groups are $\text{Homeo}^+(\mathbb{S}^3)$ and $\text{Diff}^+(\mathbb{S}^4)$, a deeper analysis of finite subgroups is required.

Proof of Theorem 2. Let $g \geq 3$ and let $\phi : \text{Mod}(S_g) \rightarrow \text{Homeo}^+(\mathbb{S}^2)$ be a homomorphism. It is a classical result of Brouwer, Eilenberg, and de Kerékjártó [5, 8, 9] that every finite subgroup of $\text{Homeo}^+(\mathbb{S}^2)$ is conjugate to a finite subgroup of $\text{SO}(3)$. These are the cyclic groups C_n , the dihedral groups D_n , and the tetrahedral, octahedral, and icosahedral groups A_4 , Σ_4 , and A_5 . This classification goes back to the work of Klein; see [6, Chapter 19] or [3, Sect.I.3.4] for treatments.

When $g \geq 3$, $\text{Mod}(S_g)$ contains a finite subgroup that is not isomorphic to any subgroup of $\text{SO}(3)$. For instance, we have the Accola–Maclachlan subgroup $C_2 \times C_{2g+2}$ of $\text{Mod}(S_g)$, which attains the $4g + 4$ bound on its largest abelian subgroup (see, for instance, [30]). This implies that ϕ has a nontrivial periodic element in its kernel. If this element is not a hyperelliptic involution, Theorem 3 implies that ϕ is trivial. Otherwise, ϕ factors through the symplectic representation. Since $\text{SO}(3)$ does not contain $(C_3)^g$ as a subgroup for $g \geq 3$, Lemma 10 implies that ϕ is again trivial.

We next consider homomorphisms to $\text{Homeo}^+(\mathbb{R}^3)$. Zimmerman showed that every orientation-preserving action by homeomorphisms of a finite group on \mathbb{S}^3 that has a global fixed point is a finite subgroup of $\text{SO}(3)$; he therefore also concludes that every finite subgroup of $\text{Homeo}^+(\mathbb{R}^3)$ is a finite subgroup of $\text{SO}(3)$ [32, Corollary 1]. This was also shown independently by Kwasik–Sun [15]. Therefore we may conclude just as we did for $\text{Homeo}^+(\mathbb{S}^2)$ above that ϕ is trivial.

Next we consider homomorphisms to $\text{Homeo}^+(\mathbb{S}^3)$. By recent work of Pardon, every continuous action of a finite group on a smooth 3-manifold is a uniform limit of smooth actions [26]. Pardon’s result guarantees that the isomorphism types of the finite subgroups of $\text{Homeo}^+(\mathbb{S}^3)$ are identical to those of $\text{Diff}^+(\mathbb{S}^3)$. By the Geometrization Theorem, every finite group acting smoothly or locally linearly on \mathbb{S}^3 is geometric, that is, it is conjugate to a finite subgroup of $\text{SO}(4)$. Every finite subgroup of $\text{SO}(4)$ is a subgroup of some $P_1 \times_{C_2} P_2$, where each P_i is one of the binary polyhedral groups: the cyclic groups C_{2n} , the binary dihedral groups D_n^* , and the binary tetrahedral, binary octahedral, and binary icosahedral groups A_4^* , Σ_4^* , and A_5^* . These facts are expounded in a survey article by Zimmermann [31], and a list of subgroups of the binary groups is given in the Appendix of [18]. (Note that $D_n^* \cong DC_n$.)

We must therefore produce for each $g \geq 42$ a finite subgroup G_g of $\text{Mod}(S_g)$ that is not a subgroup of any $P_1 \times_{C_2} P_2$. (Note that the Accola–Maclachlan subgroup $C_2 \times C_{2g+2}$ no longer suffices.) Producing such subgroups proves the theorem, for then either a nontrivial nonhyperelliptic periodic element is in the kernel of ϕ , so that ϕ is trivial; otherwise ϕ factors through the symplectic representation, and since the group $(C_3)^g$ for $g \geq 3$ is not a subgroup of any $P_1 \times_{C_2} P_2$, we conclude by Lemma 10

that ϕ is trivial. Indeed, the finite abelian subgroups of $\text{Diff}^+(\mathbb{S}^3)$ can be written as the direct product of at most two finite cyclic groups.

For $g \geq 42$, we make take G_g to be the split metacyclic group $D_{3,7}$. The split metacyclic group of order pq with p and q prime is the group with presentation

$$(3) \quad D_{p,q} = \langle a, b \mid a^q = b^p = 1, bab^{-1} = a^r \rangle$$

where r is a solution (other than 1) to the congruence $r^p \equiv 1 \pmod{q}$. Such a solution exists exactly when q is 1 mod p , and different solutions yield isomorphic groups. Note that $D_{2,q}$ is a dihedral group. Weaver computed the stable upper genus of all split metacyclic groups, and in particular he showed that $\text{Mod}(S_g)$ contains $D_{3,7}$ for all $g \geq 42$ [29, Corollary 4.8]. On the other hand, $D_{3,7}$ is not a subgroup of $\text{Diff}^+(\mathbb{S}^3)$, since the only finite subgroup of $\text{SO}(4)$ of odd order are abelian. The classification of finite subgroups of $\text{SO}(4)$ goes back to work of Seifert–Threlfall [27, 28]; see the paper of Mecchia–Seppia for a contemporary treatment [23].

Finally, let $\phi : \text{Mod}(S_g) \rightarrow \text{Diff}^+(\mathbb{S}^4)$ be a homomorphism. A theorem of Mecchia–Zimmermann [24] states that every finite group that acts smoothly on \mathbb{S}^4 and preserves orientation lies on the following list:

- (1) orientation-preserving finite subgroups of $\text{O}(3) \times \text{O}(2)$ and of $\text{O}(4) \times \text{O}(1)$,
- (2) orientation-preserving subgroups of the Weyl group $W = (C_2)^5 \rtimes \Sigma_5$,
- (3) $A_5, \Sigma_5, A_6, \Sigma_6$, and
- (4) finite subgroups of $\text{SO}(4)$ and 2-fold extensions thereof.

Since the group $(C_3)^g$ is not on this list for $g \geq 42$, to show that ϕ is trivial it suffices to show that $D_{3,7}$ is also not on this list. We first claim that $D_{3,7}$ does not lie in (1) on the list. For since $D_{3,7}$ is a semidirect product with prime-order non-commuting generators, we conclude that any isomorphism to a direct product restricts to an isomorphism on a factor. But $D_{3,7}$ is not a subgroup of any of $\text{SO}(4)$, $\text{SO}(3)$, $\text{SO}(2)$ or $\text{SO}(1)$, and since G has odd order it is also not a subgroup of the corresponding degree 2 extensions $\text{O}(4)$, $\text{O}(3)$, $\text{O}(2)$ or $\text{O}(1)$.

The group G does not lie within (2) on the list above, since it contains an element of order 7. It is not isomorphic to any of the four permutation groups within (3) on the list. Finally, G is not a 2-fold extension of any group, since it has odd order; it therefore does not lie within (4) on the list above. \square

Weaver additionally shows how to compute the genus spectra of all split metacyclic groups [29, Theorem 4.7]. Using his formulas, it is straightforward to compute that $\text{Mod}(S_g)$ contains at least one of $D_{3,7}, D_{3,13}, D_{3,19}, D_{3,31}$, or $D_{3,37}$ as a subgroup for all $g \geq 3$ except for the following values:

$$(4) \quad \{4, 5, 7, 11, 13, 16, 23, 25, 34, 41\}$$

Therefore for all homomorphisms from $\text{Mod}(S_g)$ to either $\text{Homeo}^+(\mathbb{S}^3)$ or $\text{Diff}^+(\mathbb{S}^4)$ are trivial for all $g \geq 3$ except perhaps for these ten values of g . It is possible that further analysis of finite subgroups would allow for a sharpening of the lower bound of 42 in the statement of the theorem.

REFERENCES

- [1] J. Aramayona, C. J. Leininger, and J. Souto. Injections of mapping class groups. *Geom. Topol.*, 13(5):2523–2541, 2009.
- [2] J. Aramayona and J. Souto. Homomorphisms between mapping class groups. *Geom. Topol.*, 16(4):2285–2341, 2012.
- [3] T. Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.
- [4] S. A. Broughton. Classifying finite group actions on surfaces of low genus. *J. Pure Appl. Algebra*, 69(3):233–270, 1991.
- [5] L. E. J. Brouwer. Über die periodischen Transformationen der Kugel. *Math. Ann.*, 80(1):39–41, 1919.
- [6] W. Burnside. *Theory of groups of finite order*. Dover Publications, Inc., New York, 1955. 2nd ed.
- [7] L. Chen. On the non-realizability of braid groups by homeomorphisms. <https://arxiv.org/abs/1808.08248>, 2018. to appear in *Geom. Top.*
- [8] B. de Kerékjártó. Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche. *Math. Annalen*, 80(3–7), 1919.
- [9] S. Eilenberg. Sur les transformations périodiques de la surface de sphère. *Fund. Math.*, 22:28–41, 1934.
- [10] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [11] B. Farb and H. Masur. Superrigidity and mapping class groups. *Topology*, 37(6):1169–1176, 1998.
- [12] J. Franks and M. Handel. Triviality of some representations of $MCG(S_g)$ in $GL(n, \mathbb{C})$, $\text{Diff}(S^2)$ and $\text{Homeo}(\mathbb{T}^2)$. *Proc. Amer. Math. Soc.*, 141(9):2951–2962, 2013.
- [13] W. J. Harvey and M. Korkmaz. Homomorphisms from mapping class groups. *Bull. London Math. Soc.*, 37(2):275–284, 2005.
- [14] N. V. Ivanov and J. D. McCarthy. On injective homomorphisms between Teichmüller modular groups. I. *Invent. Math.*, 135(2):425–486, 1999.
- [15] S. Kwasik and F. Sun. Topological symmetries of \mathbb{R}^3 . *Q. J. Math.*, 70(1):201–224, 2019.
- [16] J. Lanier and D. Margalit. Normal generators for mapping class groups are abundant. <https://arxiv.org/abs/1805.03666>.
- [17] C. J. Leininger and D. Margalit. On the number and location of short geodesics in moduli space. *J. Topol.*, 6(1):30–48, 2013.
- [18] D. Lima Gonçalves and J. Guaschi. *The classification of the virtually cyclic subgroups of the sphere braid groups*. SpringerBriefs in Mathematics. Springer, Cham, 2013.
- [19] K. Mann and M. Wolff. Rigidity of mapping class group actions on S^1 . <https://arxiv.org/pdf/1808.02979>, 08 2018.
- [20] V. Markovic. Realization of the mapping class group by homeomorphisms. *Invent. Math.*, 168(3):523–566, 2007.
- [21] C. L. May and J. Zimmerman. Groups of small strong symmetric genus. *J. Group Theory*, 3(3):233–245, 2000.
- [22] C. L. May and J. Zimmerman. There is a group of every strong symmetric genus. *Bull. London Math. Soc.*, 35(4):433–439, 2003.
- [23] M. Mecchia and A. Seppi. Fibered spherical 3-orbifolds. *Rev. Mat. Iberoam.*, 31(3):811–840, 2015.
- [24] M. Mecchia and B. Zimmermann. On finite groups acting on homology 4-spheres and finite subgroups of $SO(5)$. *Topology Appl.*, 158(6):741–747, 2011.
- [25] J. Müller and S. Sarkar. A structured description of the genus spectrum of abelian p -groups. *Glasg. Math. J.*, 61(2):381–423, 2019.
- [26] J. Pardon. Smoothing finite group actions on three-manifolds. <https://arxiv.org/abs/1808.08248>, 2020. to appear in *Duke Math. J.*
- [27] W. Threlfall and H. Seifert. Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes. *Math. Ann.*, 104(1):1–70, 1931.
- [28] W. Threlfall and H. Seifert. Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (Schluß). *Math. Ann.*, 107(1):543–586, 1933.
- [29] A. Weaver. Genus spectra for split metacyclic groups. *Glasg. Math. J.*, 43(2):209–218, 2001.

- [30] A. Weaver. Automorphisms of surfaces. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 257–275. Amer. Math. Soc., Providence, RI, 2002.
- [31] B. P. Zimmermann. On finite groups acting on spheres and finite subgroups of orthogonal groups. *Sib. Èlektron. Mat. Izv.*, 9:1–12, 2012.
- [32] B. P. Zimmermann. On topological actions of finite groups on S^3 . *Topology Appl.*, 236:59–63, 2018.

LEI CHEN, DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, 285 LINDE HALL, PASADENA, CA 91125, CHENLEI1991919@GMAIL.COM

JUSTIN LANIER, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, 686 CHERRY ST., ATLANTA, GA 30332, JLANIER8@GATECH.EDU