CURVE GRAPHS OF SURFACES WITH FINITE-INVARIANCE INDEX 1

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Abstract. In this note we prove a conjecture of Durham–Fanoni–Vlamis, showing that every infinite-type surface with finite-invariance index 1 fails to have a good curve graph, that is, a connected graph where vertices represent homotopy classes of simple closed curves and where the natural mapping class group action has infinite diameter orbits. Our arguments use tools developed by Mann–Rafi to study the coarse geometry of big mapping class groups.

1. Introduction

For the purposes of this note, surfaces are connected orientable 2-manifolds and curves are homotopy classes of essential simple closed curves. A surface S is of finite or infinite type according to whether or not $\pi_1(S)$ is finitely generated. The mapping class group $\mathrm{Map}(S)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S.

A central object in the study of finite-type surfaces and mapping class groups is the curve complex $\mathcal{C}(S)$, which was introduced by Harvey [6]. The vertices of $\mathcal{C}(S)$ represent curves on S and sets of vertices that have disjoint representatives bound simplices. The curve complex is flag and it is also referred to as the curve graph. Many variants of $\mathcal{C}(S)$ have also been considered—for example, by replacing curves with subsurfaces or by considering only separating or nonseparating curves.

In the finite-type setting, analyzing the natural action of $\mathrm{Map}(S)$ on $\mathcal{C}(S)$ has proved to be a useful tool in understanding $\mathrm{Map}(S)$. When S is of finite type, $\mathcal{C}(S)$ has infinite diameter when equipped with the path metric, and its geometry is Gromov hyperbolic, as shown by Masur–Minsky [9]. Further, the orbits of this action have infinite diameter, and analyzing these orbits gives information about the Nielsen–Thurston type of a mapping class.

For any infinite-type surface S, it is not hard to see that $\mathcal{C}(S)$ has diameter 2. If we have any hope of recovering the interesting mapping class group actions found in the finite-type setting, it is necessary to build alternatives to $\mathcal{C}(S)$. Some authors have built alternatives by replacing curves with other objects, as Bavard has done using rays [4]. In this note, following Durham–Fanoni–Vlamis, we consider alternatives where vertices correspond to curves on S, but where edges need not correspond to disjointness [5]. We will say that a graph Γ is a curve graph for S if the vertices of Γ represent some (not necessarily proper) subset of curves on S and Γ admits an action that restricts to the natural action of $\mathrm{Map}(S)$ on vertices. Whenever a curve graph Γ for S is connected and has an infinite diameter orbit under the natural action of $\mathrm{Map}(S)$, we will call Γ a good curve graph for S.

Durham–Fanoni–Vlamis defined an invariant of infinite-type surfaces that in most cases determines whether or not S has a good curve graph [5]. The finite-invariance index $\mathfrak{f}(S)$ of a surface is the size of the largest finite $\mathrm{Map}(S)$-invariant collection of disjoint closed proper subsets of ends of S; see Section 2 for more details. Using this invariant, Durham–Fanoni–Vlamis were able to characterize in most cases whether S has a good curve graph.

Theorem 1.1 (Durham-Fanoni-Vlamis)**.** *If $\mathfrak{f}(S) \geq 4$, then $\mathrm{Map}(S)$ admits an unbounded action on a graph consisting of curves. If $\mathfrak{f}(S) = 0$, then $\mathrm{Map}(S)$ admits no such unbounded action.*

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Aramayona–Valdez also characterized whether certain classes of infinite-type surfaces have candidates for good curve graphs [2]. Their results break into two cases, depending on whether S has finite or infinite genus. We summarize their relevant results as follows.

Theorem 1.2 ([2], Theorems 1.4 and 1.7)**.** *Let S be an infinite-type surface.*

(1) If S has finite genus and no isolated punctures, then a $\mathrm{Map}(S)$-invariant subgraph $\mathcal{G}(S)$ has infinite diameter if and only if it contains no separating curves that cut off a disk containing some, but not all, of the punctures.

(2) If S is a blooming Cantor tree and $\mathcal{G}(S)$ is a $\mathrm{Map}(S)$-invariant subgraph of $\mathcal{C}(S)$, then $diam(\mathcal{G}(S)) = 2$.

The statement given in *(2)* is a correction of the statement given by Aramayona–Valdez [2, Theorem 1.7]; their original statement contradicts Theorem 1.1. The proof that they give assumes not only that S has infinite genus, but that every end is accumulated by genus [1]; this further hypothesis implies that S is a blooming Cantor tree. Note that *(2)* overlaps with the result proven by Durham–Fanoni–Vlamis, since the blooming Cantor tree has finite-invariance index 0. Observe that the set of surfaces described in *(1)* by Aramayona–Valdez all have finite-invariance index 0 or ∞, but there is no requirement in this result for the natural action of $\mathrm{Map}(S)$ to have infinite diameter orbits.

In their paper, Durham–Fanoni–Vlamis make some observations about the cases of finite-invariance index 1, 2, and 3. For $\mathfrak{f}(S) = 2$ and $\mathfrak{f}(S) = 3$, they give examples showing that finite-invariance index is too coarse an invariant to determine whether S has a good curve graph. On the other hand, they conjecture that $\mathfrak{f}(S) = 1$ implies that S has no good curve graph [5, Conjecture 9.1].

The main result of this note confirms their conjecture.

Theorem 1.3. *Let S be a surface of infinite type with $\mathfrak{f}(S) = 1$. Then S does not have a good curve graph.*

We give two proofs of this theorem. We begin by proving some preliminary lemmas, squaring up definitions by employing a partial order on the ends of a surface defined by Mann–Rafi. Theorem 1.3 then follows directly from a strictly stronger result of Mann–Rafi [8, Proposition 3.1]. We apply this approach in Section 3. Our second proof uses the same preliminary lemmas but then follows the style of the arguments of Durham–Fanoni–Vlamis. We apply this approach in Section 4.

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2. Big surfaces, finite-invariance index, and coarse boundedness

The goal of this section is to briefly overview the aspects of infinite-type surfaces relevant to our work. In particular, we review the classification of infinite type surfaces as well as notions related to the finite-invariance index, self-similarity of a space of ends, and coarse boundedness of a group. For more comprehensive treatments of these topics, we refer the reader to [10], [5], [8], and [3].

Classifying big surfaces. The classification of infinite-type surfaces was first given by Kerékjártó and was clarified and extended by Richards [7, 10].

The classification implies that every connected orientable 2-manifold may be constructed as follows. Begin with a sphere. Puncture it, without loss of generality, along some closed subset of a Cantor set. This set of punctures is denoted $E(S)$, and punctures are called ends. Finally, add handles to the surface so that the only accumulation points of sequences of handles are in $E(S)$. This marks some closed subset of $E(S)$ as being accumulated by genus. This space of genus ends is denoted $E^g(S)$ and is recorded as a subspace of $E(S)$. Note that $E^g(S)$ is also a closed subset of a Cantor set. Observe that closed surfaces and punctured surfaces of finite type are special cases of this construction.

Let $\mathrm{g}(S) \in \mathbb{N} \cup \{\infty\}$ equal the genus of S. The classification states that the triple $(\mathrm{g}(S), E(S), E^g(S))$ uniquely determines S up to homeomorphism. For more details on the classification and on the definition of the spaces of ends, see [5].

The finite-invariance index. Following Durham–Fanoni–Vlamis, we make the following definitions. We say that a collection $\mathcal{P}$ of disjoint subsets of the space of ends is $\mathrm{Map}(S)$-invariant if for every $P \in \mathcal{P}$ and for every $\varphi \in \mathrm{Map}(S)$ there exists $Q \in \mathcal{P}$ such that $\varphi(P) = Q$. The finite-invariance index of S, denoted $\mathfrak{f}(S)$, is defined as follows:

- $\mathfrak{f}(S) \geq n$ if there is a $\mathrm{Map}(S)$-invariant collection $\mathcal{P}$ of disjoint closed proper subsets of $E(S)$ satisfying $|\mathcal{P}| = n$;
- $\mathfrak{f}(S) = \infty$ if $\mathrm{g}(S)$ is finite and positive;
- $\mathfrak{f}(S) = 0$ otherwise.

We say that $\mathfrak{f}(S) = n$ if $\mathfrak{f}(S) \geq n$ but $\mathfrak{f}(S) \not\geq n+1$.

For any $\mathrm{Map}(S)$-invariant collection $\mathcal{P}$ of disjoint closed proper subsets of $E(S)$, call the elements of $\mathcal{P}$ finite-invariance sets. When $\mathcal{P}$ contains only one set, we also call $\mathcal{P}$ a finite-invariance set.

A partial order on the space of ends and self-similarity. Following Mann–Rafi, we make the following definitions. For $x, y \in E(S)$, we say $x \preccurlyeq y$ if every clopen neighborhood of y contains a homeomorphic copy of some clopen neighborhood of x. We say x and y are equivalent if $x \preccurlyeq y$ and $y \preccurlyeq x$. We will make use of the following result concerning this partial order.

Proposition 2.1 ([8], Proposition 4.7). *The partial order $\preccurlyeq$ has maximal elements. Furthermore, the equivalence class of every maximal element is either finite or a Cantor set.*

We let $\mathcal{M}(E)$ denote the set of equivalence classes of maximal ends of $E(S)$.

Continuing to follow Mann–Rafi, a pair (E, E^g) is self-similar if for any decomposition $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ of E into pairwise disjoint clopen sets, there exists a clopen set D contained in some E_i such that the pair $(D, D \cap E^g)$ is homeomorphic to (E, E^g).

A Polish group G is (globally) coarsely bounded, or CB, if every compatible left-invariant metric on G gives G finite diameter. Mann–Rafi show that self-similarity of the space of ends of a surface S is a sufficient condition for $\mathrm{Map}(S)$ to be CB.

Proposition 2.2 ([8], Proposition 3.1). *Let S be a surface of infinite or zero genus. If the space of ends of S is self-similar, then* $\mathrm{Map}(S)$ *is CB.*

Good curve graphs vs. coarse boundedness. If a surface S has a good curve graph, then $\mathrm{Map}(S)$ is not CB. However, the converse is not true. There are examples of surfaces whose mapping class groups are not CB, and so admit an action with an unbounded orbit on some metric space, yet do not admit such an action on any metric space arising as a connected complex of curves.

An example of such a surface is the tripod surface T, which has exactly three ends, all accumulated by genus. By Proposition 9.2 of [5], T has no good curve graph. However, T has a nondisplaceable subsurface, and so $\mathrm{Map}(T)$ is not CB by Theorem 1.9 of [8].

3. Main theorem: first approach

In this section we begin by relating the finite-invariance index, the partial order on ends, and self-similarity in the case $\mathfrak{f}(S) = 1$. We then give our first proof of Theorem 1.3.

Lemma 3.1. *Let S be a surface of infinite type with $\mathfrak{f}(S) = 1$. Then $\mathcal{M}(E)$ consists of a unique equivalence class that is either a point or a Cantor set.*

Proof. Let S be a surface with $\mathfrak{f}(S) = 1$ and with end space E. It follows that $\mathcal{M}(E)$ consists of a unique equivalence class, for otherwise two of the equivalence classes could be taken as a collection of finite-invariance sets for S, as these would be closed, $\mathrm{Map}(S)$-invariant, proper, and disjoint, contradicting $\mathfrak{f}(S) = 1$.

We know by Proposition 2.1 that an individual maximal equivalence class is either finite or a Cantor set. If $\mathcal{M}(E)$ is finite, then it must consist of a single point because $\mathfrak{f}(S) = 1$, just as before. Therefore $\mathcal{M}(E)$ is either a point or a Cantor set. □

Lemma 3.2. *Let S be a surface of infinite type with $\mathfrak{f}(S) = 1$. Then the set of ends of S is self-similar.*

Proof. Let (E, E^g) be the set of ends of S and consider any decomposition $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ into pairwise disjoint clopen sets. Since S is of infinite type and $\mathfrak{f}(S) = 1$, by Lemma 3.1 we have that $\mathcal{M}(E)$ is either a point or a Cantor set. Consider $\mathcal{M}(E) \cap (E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n)$. If $\mathcal{M}(E)$ is a point, then there is one set in the decomposition, say E_i, that contains $\mathcal{M}(E)$. If we let $D = E_i$, then $(D, D \cap E^g) \cong (E, E^g)$. This homeomorphism exists by the classification of surfaces, since D is a clopen subset of E and $\mathcal{M}(E)$ is the unique maximal equivalence class in E. If $\mathcal{M}(E)$ is a Cantor set, we pick any set in the decomposition, say E_i, that has non-empty intersection with $\mathcal{M}(E)$; this intersection is therefore homeomorphic to $\mathcal{M}(E)$. Once again setting $D = E_i$, we have that $(D, D \cap E^g) \cong (E, E^g)$, with the same justification. So in both cases, (E, E^g) is self-similar. □

Proof #1 of Theorem 1.3. Since $\mathfrak{f}(S) = 1$, $\mathrm{g}(S) = 0$ or ∞. By Lemma 3.2, the set of ends of S is self-similar. By Proposition 2.2, this implies that $\mathrm{Map}(S)$ is CB, and so it has no unbounded length function. In particular, it cannot act with infinite diameter orbits on a connected curve graph. We conclude that S has no good curve graph. □

4. Main theorem: second approach

In this section we will first review the approach introduced by Durham–Fanoni–Vlamis to show that a surface has no good curve graph. We then extend their approach under the hypothesis $\mathfrak{f}(S) = 1$. We conclude with our second proof of Theorem 1.3.

Durham–Fanoni–Vlamis introduced the following tool for showing that a curve graph is not good.

Proposition 4.1 ([5], 4.1)**.** *Let S be an oriented surface and $\Gamma = \Gamma(S)$ be a connected graph consisting of curves on which the mapping class group acts. Let $\mathcal{V} \subset \Gamma \times \Gamma$ satisfying:*

1. *there exists a vertex $c \in \Gamma$ such that, up to the action of $\mathrm{Map}(S)$, there is a finite number of pairs $(a, b) \in \mathcal{V}$ with $a, b \in \mathrm{Map}(S) \cdot c$, and*
2. *for every $a, b \in \mathrm{Map}(S) \cdot c$ with $(a, b) \notin \mathcal{V}$, there exists $d \in \mathrm{Map}(S) \cdot c$ such that (a, d) and (b, d) belong to $\mathcal{V}$.*

Then every $\mathrm{Map}(S)$-orbit in Γ has finite diameter.

Durham–Fanoni–Vlamis used this tool to derive three easily-applied criteria for showing that a curve graph is not good. They used these criteria to show that surfaces with finite-invariance index 0 do not have good curve graphs. These three criteria are given in the following proposition.

Proposition 4.2 ([5], 4.2)**.** *Suppose that $\Gamma = \Gamma(S)$ is a connected graph consisting of curves with an action of $\mathrm{Map}(S)$ and:*

1. *S has infinitely many isolated punctures and Γ contains a vertex bounding a finite-type genus-0 surface, or*
2. *$\mathrm{genus}(S) = \infty$, S has either no punctures or infinitely many isolated punctures, and Γ contains a curve bounding a finite-type surface, or*
3. *$\mathrm{genus}(S) = \infty$ and Γ contains a nonseparating curve,*

then $\mathrm{Map}(S)$ *acts on* $\Gamma(S)$ *with finite-diameter orbits.*

Durham–Fanoni–Vlamis show that separating curves are bad when they cut off a finite-type surface from an "infinite pool" of genus and/or isolated punctures. In Propositions 4.3 and 4.4 we show that, under the hypothesis of $\mathfrak{f}(S) = 1$, separating curves are also bad when they cut off an infinite-type surface and still leave behind an "infinite pool" of what has been cut off. For two distinct disjoint separating curves a and b, we say that a is nested in b with respect to $E' \subset E(S)$ if b separates a from E'. If a and b are two distinct disjoint separating curves and neither is nested in the other with respect to E', we say that a and b are unnested with respect to E'.

From Proposition 3.1, we have that when $\mathfrak{f}(S) = 1$, $\mathcal{M}(E)$ contains a unique equivalence class this is either a point or a Cantor set. We treat these two cases in the following two propositions.

Proposition 4.3. *Suppose* S *is of infinite type with* $\mathfrak{f}(S) = 1$ *and that* Γ *is a connected curve graph for* S*. Suppose further that the unique equivalence class in* $\mathcal{M}(E)$ *is a point. Then* $\mathrm{Map}(S)$ *acts on* Γ *with finite-diameter orbits.*

Proof. Let S and Γ be as in the statement and let e be the point in the unique equivalence class in $\mathcal{M}(E)$. Let c be a curve in Γ. Since $\mathfrak{f}(S) = 1$, then $\mathrm{g}(S) = 0$ or ∞. If c is a nonseparating curve, then $\mathrm{g}(S) = \infty$, Proposition 4.2(3) applies, and we are done.

If c is separating and cuts off a finite-type surface, then either Proposition 4.2(1) or (2) applies, as follows. Since we already have that $\mathrm{g}(S) = 0$ or ∞, then it suffices to show that S has either 0 or infinitely-many isolated punctures. The surface S cannot have two or more (but finitely-many) isolated punctures, since these would yield finite-invariance sets and contradict $\mathfrak{f}(S) = 1$. If there is a single isolated puncture, then it must be maximal and so equal e. But then S is the plane and so not of infinite type.

Otherwise, c is separating but does not cut off a finite-type subsurface. Set

$$\mathcal{V} = \{(a, b) \mid i(a, b) = 0,\ a \text{ and } b \text{ are unnested with respect to } e\}.$$

By the classification of surfaces and the self-similarity of $E(S)$, there is a unique pair of separating curves $(a, b) \in \mathcal{V}$ with $a, b \in \mathrm{Map}(S) \cdot c$, up to the action of $\mathrm{Map}(S)$. Therefore condition (1) of Proposition 4.1 is satisfied.

Next, consider $a, b \in \mathrm{Map}(S) \cdot c$ such that $(a, b) \notin \mathcal{V}$. This means that either a and b intersect, one is nested in the other with respect to e, or they are not distinct. In each case we will find $d \in \mathrm{Map}(S) \cdot c$ such that (a, d) and (b, d) belong to $\mathcal{V}$. If a and b intersect, then together they fill a finite-type subsurface F. By Lemma 3.2, one of the components of $S \setminus F$ has end space homeomorphic to S. Thus, we can pick a separating curve d in this component that is in $\mathrm{Map}(S) \cdot c$ and is disjoint from and unnested with each of a and b with respect to e. Therefore, $(a, d), (b, d) \in \mathcal{V}$. Similarly, if a and b are nested with respect to e, there is again a component of $S \setminus \{a \cup b\}$ with end space homeomorphic to S where we may pick the desired curve d that is disjoint from and unnested with each of a and b. A similar argument applies if a and b are not distinct. Therefore condition (2) of Proposition 4.1 is satisfied and we have shown that $\mathrm{Map}(S)$ acts on Γ with finite-diameter orbits. □

We now prove the analogous result to Proposition 4.3 for surfaces with $\mathfrak{f}(S) = 1$ where the unique equivalence class in $\mathcal{M}(E)$ is a Cantor set.

Proposition 4.4. *Suppose* S *is of infinite type with* $\mathfrak{f}(S) = 1$ *and that* Γ *is a connected curve graph for* S*. Suppose further that the unique equivalence class in* $\mathcal{M}(E)$ *is a Cantor set. Then* $\mathrm{Map}(S)$ *acts on* Γ *with finite-diameter orbits.*

Proof. Let S and Γ be as in the statement and let $\mathcal{C}$ be the Cantor set in the unique equivalence class in $\mathcal{M}(E)$. Let c be a curve in Γ. If c is a nonseparating curve or if c separates off a finite-type subsurface, we can conclude that Γ is not good by Proposition 4.2. This follows by the same arguments as in Proposition 4.3.

Otherwise, c is separating and both components of $S \setminus c$ are of infinite type. We consider the two cases: either only one component of $S \setminus c$ has end space with nontrivial intersection with $\mathcal{C}$, or instead both do. In the former case, the same argument holds as in Proposition 4.3, substituting $\mathcal{C}$ for e.

The remaining case follows similarly to the treatment of the Cantor tree surface by Durham–Fanoni–Vlamis [5, Theorem 1]. Since S has a unique maximal equivalence class of ends, any clopen neighborhood in $E(S)$ of any maximal end is homeomorphic to $E(S)$. In particular, the end spaces of the two components of $S \setminus c$ are each homeomorphic to $E(S)$. Set

$$\mathcal{V} = \{(a, b) \mid i(a, b) = 0\}.$$

By the classification of surfaces and the self-similarity of $E(S)$, there is a unique pair of distinct disjoint separating curves $(a, b) \in \mathcal{V}$ with $a, b \in \mathrm{Map}(S) \cdot c$, up to the action of $\mathrm{Map}(S)$. Therefore condition (1) of Proposition 4.1 is satisfied.

Next, consider $a, b \in \mathrm{Map}(S) \cdot c$ such that $(a, b) \notin \mathcal{V}$. Then a and b intersect. In the former case, a and b and together they fill a finite-type subsurface F. At least one complementary component of $S \setminus F$ has end space homeomorphic to $E(S)$ by Lemma 3.2. We can therefore pick a separating curve d in this complementary component that is in $\mathrm{Map}(S) \cdot c$ and that is disjoint from each of a and b. We then have $(a, d), (b, d) \in \mathcal{V}$. Condition (2) of Proposition 4.1 is therefore satisfied and we have shown that $\mathrm{Map}(S)$ acts on Γ with finite-diameter orbits. □

Proof #2 of Theorem 1.3. By Lemma 3.1 $\mathcal{M}(E)$ is either a point or a Cantor set. If it is a point, we apply Proposition 4.3. If it is a Cantor set, we apply Proposition 4.4. We conclude that S has no good curve graph. □

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